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INTEGRABLE PERTURBATIONS OF A BIRKHOFF BILLIARD INSIDE AN ELLIPSE†

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Integrable perturbations which depend only on the coordinates of a Birkhoff billiard inside an ellipse are considered. The class of integrable potentials which are polynomials in x, y, x^{-1} and y^{-1} is described completely.

1. STATEMENT OF THE PROBLEM

A Birkhoff billiard normally describes a particle which moves freely inside a certain region in a plane. On colliding with the boundary of that region, which is convex, there is absolutely elastic impact with equal angles between the path and the boundary before and after impact. We will call the behaviour on the boundary the *billiard law*.

There are a number of integrable billiard problems (cf. [1, 2] and the references therein). The best known of these is the problem inside an ellipse. Integrable perturbations of this and related integrable problems (geodesics on an ellipsoid, Neumann's problem and Kepler's problem) were studied even by the classics [3]. Interest in these problems has recently been revived [4–9].

We shall discuss the question of the integrability of problems of the motion of a particle inside an ellipse under the action of a potential force when the behaviour on the boundary corresponds to a billiard law. Kozlov [4] has introduced the family of integrable potentials

$$V = k(x^2+y^2)+\frac{\alpha}{x^2}+\frac{\beta}{y^2}+\frac{\gamma_1}{r_1}+\frac{\gamma_2}{r_2}, \quad \alpha,\ \beta,\ \gamma_i \in R$$

where r_i are the distances to the foci of the ellipse. Using Kozlov's ideas, this class of potentials can be enlarged considerably by the method described below. We will obtain two families of potentials: (1) in the form of polynomials, (2) in the form of Laurent polynomials. Although polynomial potentials have arisen in similar problems before [10, 11], potentials in the form of Laurent polynomials are basically new and have a non-trivial dynamics. With the proposed method we can find all the integrable potentials of a certain type in explicit form, and also prove that there are no other potentials in that class.

2. ONE-PARAMETER SOLUTIONS

The problem of the free motion of a particle inside an ellipse with semi-axes a and b and a billiard law on the boundary has the well-known complementary integral

$$F_0 = \frac{\dot{x}^2}{a}+\frac{\dot{y}^2}{b}-\frac{(\dot{x}y-x\dot{y})^2}{ab}$$

Kozlov's method involved finding potentials V for which there is a complementary integral of the form $F = F_0 + f$, where the function f depends only on the coordinates. Choosing a perturbation which depends only on the coordinates ensures that F is conserved on impact at the boundary. The requirement that the value of F shall also be conserved inside the ellipse (where the equations of motion are $\ddot{x} = -V_x, \ddot{y} = -V_y$) yields a consistency condition for V, which can be described by the following equation

$$\lambda V_{xy}+3(yV_x-xV_y)+(y^2-x^2)V_{xy}+xy(V_{xx}-V_{yy})=0, \quad \lambda=a-b \tag{2.1}$$

We will seek solutions of Eq. (2.1) in the form of Laurent polynomials in x, y, that is

$$V(x,\ y,\ \lambda)=\sum_{s\geq m,n\geq k} a_{m,n}(\lambda)x^m y^n, \quad k,\ s\in Z \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain a system of difference equations

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$$\lambda mna_{n,m} = (n+m)(ma_{n-2,m} - na_{n,m-2}) \tag{2.3}$$

It will be assumed below that $\lambda \neq 0$. We will take the degree of an element $a_{m,n}$ to mean the sum $m + n$. The following lemma follows from (2.3).

Lemma 1. If $a_{m_0 n_0}$ is an element of the lowest non-zero degree, then $n_0 = 0$ or $m_0 = 0$.
Let m_0 denote the lowest non-zero degree. Then all the elements, apart from a_{0,m_0} and $a_{m_0,0}$, are equal to zero and at least one of the elements a_{0,m_0}, $a_{m_0,0}$ is non-zero.

Lemma 2. The elements a_{2k,m_0} can be expressed in terms of a_{0,m_0} by the formula

$$a_{2k,m_0} = \frac{(2+m_0)(3+m_0)\ldots(2k+m_0)}{(2\lambda)^k k!} a_{0,m_0}, \quad k \in N$$

Suppose that $m_0 < 0$. The following proposition follows from Lemmas 1 and 2.

Proposition 1. (a) If at least one of the elements $a_{m,n}$, $mn \neq 0$ is non-zero, then $m_0 = -2k$ for some $k \in N$;
(b) $a_{2k+1,0} = a_{0,2k+1} = 0$ for any $k \in Z$.
Let

$$V_k(x, y, \lambda, \alpha) = \sum_{i=0}^{k-2} (-1)^i \sum_{s=1}^{k-i-1} U_{kis}(x, y, \lambda, \alpha) + \alpha y^{-2k}$$

$$W_k(x, y, \lambda, \alpha) = \sum_{i=0}^{k-2} \sum_{s=1}^{k-i-1} (-1)^s U_{kis}(y, x, \lambda, \alpha) + \alpha x^{-2k}$$

$$U_{kis}(x, y, \lambda, \alpha) = \binom{s+i-1}{i} \frac{[1-(k-i)][2-(k-i)]\ldots[s-(k-i)]}{\lambda^{s+i} s!} \alpha x^{2s} y^{-2k+2i}$$

Theorem 1. For any α the functions $V_k(x, y, \lambda, \alpha)$ and $W_k(x, y, \lambda, \alpha)$ are solutions of Eq. (2.1) of the form (2.2) with non-zero element of lowest degree $a_{0,m_0} = \alpha$ and $a_{m_0,0} = \alpha$ respectively.
The proof of Theorem 1 uses the following lemma.

Lemma 3. If $a(1, n) = 1 = a(k, 1)$ and $a(n, m) = a(n-1, m) + a(n, m-1)$, then

$$a(n, m) = \binom{n+m-2}{n-1}$$

We note that $V_k(x, y, \lambda, \alpha) = W_k(y, x, -\lambda, \alpha)$ (according to the general property of Eqs (2.1): if $\varphi(x, y)$ is a solution of Eq. (2.1) with parameter λ, then $\varphi(x, y)$ is a solution of Eq. (2.1) with parameter $-\lambda$. By the potential V_1 and W_1 we must understand Kozlov potentials α/x^2 and β/y^2.

Example 1. For $V_2(x, y, \lambda) = (\lambda - x^2)/(\lambda y^4)$, the complementary integral is given by the formula

$$F = \frac{\dot{x}^2}{a} + \frac{\dot{y}^2}{b} - \frac{(\dot{x}y - \dot{y}x)^2}{ab} + \frac{2x^4 + (2b-4a)x^2 + 2y^2x^2}{\lambda aby^4}$$

The potentials described in Theorem 1 depend on one parameter—the first non-zero element $a_{-2k,0}$ or $a_{0,-2k}$. We shall call them one-parameter potentials. We will be interested in the investigation of more complicated potentials below.

3. MULTI-PARAMETER POTENTIALS

Lemma 4. If the potential V has one non-zero element among elements of the form a_0, k and $a_{l,0}$, say, $a_{0,-2k_1} = \alpha_1, \ldots, a_{0,-2k_s} = \alpha_s$, then

$$V = \sum_{i=1}^{s} V_{k_i}(x, y, \alpha_i)$$

The potentials discussed in Lemma 4 will be called s-parameter potentials.

Example 2. Consider the case of the two-parameter potential

$$W_3^2 = \alpha x^{-6} + \frac{2\alpha}{\lambda} y^2 x^{-6} + \beta x^{-4} + \frac{\alpha}{\lambda^2} y^4 x^{-6} + \frac{\beta\lambda + \alpha}{\lambda^2} y^2 x^{-4}$$

Let $A_k = \{V_k(x, y, \alpha) \,|\, \alpha \in R\}$ and $B_k = \{W_k(x, y, \alpha) \,|\, \alpha \in R\}$ denote the potential spaces in which $m_0 < 0$. There are also potentials for which $m_0 > 0$. Let C_l be the space of solutions of Eq. (2.1) which are polynomials in x, y of degree not more than $2l$. Their general term is given by the formula

$$a_{2n,\,2m} = \sum_{t=1}^{m} \frac{(-1)^{m-t}\binom{n+m-t-1}{n-1}(1+m)(2+m)\ldots(n+m)}{n!\lambda^{n+m-t}} a_{0,2t} + \sum_{t=1}^{n} \frac{(-1)^{m}\binom{n+m-t-1}{m-1}(1+n)(2+n)\ldots(m+n)}{m!\lambda^{n+m-t}} a_{2t,0}, \quad n+m \le l$$

and the initial conditions $a_{0,2t}$, $a_{2t,0}$, where $1 \le t \le l$, are determined from the system of linear equations

$$2m a_{2(l-m),\,2m} = (2(l-m)+2) a_{2(l-m)+2,\,2m+2}, \quad 0 \le m \le l-1$$

Example 3. When $l = 2$ we obtain

$$V = \beta x^2 + \alpha y^2 + \frac{2}{\lambda}(\alpha - \beta) x^2 y^2 + \frac{(\alpha - \beta)}{\lambda}(x^4 + y^4)$$

Example 4. Let $l = 3$, $a_{0,2} = \alpha$, $a_{2,0} = \beta$, $a_{4,0} = \gamma$. Then

$$a_{2,2} = \frac{2}{\lambda}(\alpha - \beta), \quad a_{0,4} = \frac{2}{\lambda}(\alpha - \beta) - \gamma$$

$$a_{0,6} = a_{6,0} = \frac{1}{\lambda^2}(\alpha - \beta) - \frac{\gamma}{\lambda}, \quad a_{4,2} = a_{2,4} = 3a_{0,6}$$

For a mechanical realization of these examples, we can consider a billiard as a system in which a particle of unit mass moves inside an ellipse in a gravitational field. The distribution of masses inside the ellipse is determined by the density ρ, which is given, by Poisson's equation, in the form

$$\rho(x, y) = \Delta V(x, y)/(4\pi)$$

Finally, we have the following theorem.

Theorem 2. The solution space R of Eq. (2.1) of the form (2.2) can be described by the formula

$$R = \oplus_{k=1}^{\infty} A_k + \oplus_{k=1}^{\infty} B_k + \bigcup_{l=1}^{\infty} C_l$$

4. CONCLUDING COMMENTS

1. We will now consider the case $\lambda = 0$.
Then system (2.3) reduces to the form

$$(n+m)(m a_{n-2,m} - n a_{n,m-2}) = 0$$

When $n = -m$ we obtain solutions of the form $V_m(x, y) = \alpha x^{-m-2} y^m$ and $W_m(x, y) = \alpha x^{m-2} y^{-m-2}$.
In other words, if $a_{m0,\,n0} \ne 0$ is an element of the lowest degree then $m_0 = 0$ (or $n_0 = 0$) and $n_0 = 2k$, $k \in N$ (or $m_0 = 2k$, $k \in N$). The general formula for $a_{n,m}$ in this case is

$$a_{2k-2s,\,2s} = \frac{k(k-1)\ldots(k-s+1)}{s!} a_{2k,\,0}, \quad s = 1, \ldots, k-1$$

2. Similar methods can be used for the problem of geodesics on an ellipsoid and the problem of billiards inside an ellipsoid.

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